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Quantum Corrections to the Spacetime Metric from Geometric Phase Space Quantization

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We consider the possibility that the physical spacetime of a quantum particle may be regarded as a four-dimensional hypersurface locally embedded in eight-dimensional phase space. We show that, as a consequence, accelerated particles are seen to live in a curved spacetime, and, in the particular case of uniform acceleration, we are led to a generalization of the Rindler metric which implies, for a uniformly accelerated particle, a discrete energy spectrum.

1. INTRODUCTION

Recently the possibility has been considered that one-particle quantum mechanics may acquire a geometric interpretation (Caianiello, 1980, 1983), through a quantization model formulated in a curved eight-dimensional manifold M_8, with coordinates $x^A = (x^\mu, (\hbar/mc)\dot{x}^\mu)$, where x^μ is the usual position four-vector, and $\dot{x}^\mu = dx^\mu/ds$ is the relativistic four-velocity (conventions: $A, B, \ldots = 1, \ldots, 8$; $\mu, \nu, \ldots = 1, \ldots, 4$).

We do not intend to discuss here in detail this model [many formal as well as physical aspects of which have been analyzed in previous papers (Caianiello, 1980, 1981, 1983; Caianiello and Vilasi, 1983)]; we only recall that in this context the position-momentum commutation rules, $[x, p] = i\hbar$, are reproduced by representing these operators as covariant derivatives with an appropriate connection in the eight-dimensional manifold M_8. In this way quantization is geometrically understood as a consequence of curvature in phase space.

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Of the various consequences of this geometric approach (Caianiello *et al.*, 1982*a*; Scarpetta, 1984; Guz and Scarpetta, 1986), we wish to focus our attention on the fact that the relativistic spacetime interval, $ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$, where $\eta = \mathrm{diag}(---+)$ is the Minkowski metric, must be replaced by a more general physical invariant (see also Brandt, 1984, 1986), representing the infinitesimal distance in M_8, i.e., $d\tau^2 = g_{AB}\, dx^A\, dx^B$. Assuming for g_{AB}, in the absence of gravity, the simplest form $g_{AB} = \eta_{\mu\nu} \otimes \eta_{\mu\nu}$ [in agreement with the uncertainty principle (Caianiello, 1984), and also with very general arguments based on entropy and information theory (Caianiello, 1986)], we are led then to consider, as the fundamental infinitesimal interval for a particle, the following eight-dimensional line element:

$$d\tau^2 = dx^\mu\, dx_\mu + \frac{\hbar^2}{m^2 c^2}\, d\dot{x}^\mu\, d\dot{x}_\mu \tag{1.1}$$

(Here we have written explicitly the dimensional constants, to stress the fact that $d\tau^2$ reduces to the usual four-dimensional distance in the classical macroscopic limit $\hbar \to 0$; henceforth we shall use, however, natural units $\hbar = c = 1$.)

Starting from the expression (1.1), one could try to formulate an eight-dimensional generalization of special relativity (Scarpetta, 1984, and general relativity (Brandt, 1984, 1986, 1987): our present aim, however, is more modest. If we accept the hypothesis that the microscopic spacetime should be regarded as a four-dimensional hypersurface locally embedded in the larger manifold M_8, as the previously quoted geometric quantization scheme suggests, it follows that the metric of the physical spacetime V_4 is locally induced by the generalized line element (1.1) of M_8, through the parametric equations which govern the embedding of V_4 into M_8. We propose to show in this paper that accelerated particles ($d\dot{x}^\mu \neq 0$) are associated to four-dimensional hypersurfaces whose curvature is in general nonvanishing: at this semiclassical level, therefore, the effective spacetime geometry experienced by interacting particles is curved.

This curvature, induced not by matter through conventional Einstein equations, but by *motion in momentum space*, as shown in Section 2, vanishes in the limit $\hbar \to 0$; it represents thus a quantum correction to the given background geometry (always flat throughout this paper) experienced in four dimensions by classical macroscopic test bodies. The crucial point is that, as a consequence, particles differently affected by external interactions, and therefore with different trajectories in momentum space, *are seen to live in different four-dimensional geometries*: this prospective realizes, already at a semiclassical level, the old conjecture that in a quantum theory of gravity the spacetime metric should be observer dependent (Gibbons and Hawking, 1977).

The simplest case one may consider, i.e., the case of uniformly accelerated particles, will be introduced as an example in Section 3. We obtain then, according to this scheme, corrections to the geometry of the Rindler manifold which induce curvature and shift the boundary of the Rindler wedge, thus suggesting modifications of the well-known relation between acceleration and temperature (Davies, 1975; Unruh, 1976).

The main results of this paper, together with the possibility of further applications of this geometric approach to quantization, will be briefly discussed in Section 4.

2. LOCAL ISOMETRIC EMBEDDING OF SPACETIME INTO PHASE SPACE

If we regard the spacetime V_4, at the microscopic level, as a four-dimensional hypersurface locally embedded in M_8, in order to obtain the four-dimensional metric [induced by (1.1)] which determines its intrinsic geometry we must give the parametric equations governing its embedding (i.e., representing V_4 as a submanifold of M_8), that is, $x^A = x^A(\xi^\mu)$, where x^A are coordinates on M_8 and ξ^μ on V_4.

To this purpose, first we observe that, according to the generalized invariant $d\tau$ defined by the eight-dimensional line element (1.1), the spacetime intervals $ds^2 = dx^\mu\, dx_\mu$ are no longer invariant, but depend on the trajectory in momentum space, that is, on the choice of $\dot{x}^\mu(s)$ (just as in special relativity the time intervals dt lose their absolute meaning and become dependent on the curve $x^i(t)$ which describes motion in three-dimensional space). For any given trajectory $\dot{x}^\mu(s)$ we have in fact $d\dot{x}^\mu = \ddot{x}^\mu\, ds$ and we obtain, from equation (1.1),

$$d\tau^2 = ds^2\left(1 - \frac{|\ddot{x}|^2}{m^2}\right) \tag{2.1}$$

where $|\ddot{x}|^2 = |\eta_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu|$ is the squared length of the (spacelike) relativistic acceleration four-vector. As $d\tau$ is now the invariant interval, motions with different accelerations clearly correspond to different values of ds. Along a classical path, defined by $x^\mu(s)$ and $\dot{x}^\mu(s)$, we have therefore a generalized definition of the invariant proper time, namely

$$d\tau = dt\,(1 - |\mathbf{v}|^2)^{1/2}\left(1 - \frac{|\ddot{x}|^2}{m^2}\right)^{1/2} \tag{2.2}$$

where $v^i = dx^i/dt$.

This argument can be applied to a classical observer, whose motion is represented by a well-defined trajectory; for a quantum particle, however, the notion of trajectory is meaningless, since position and momentum cannot be simultaneously defined with arbitrary high precision. The motion of a quantum particle therefore cannot be geometrically represented by a unidimensional world line $x^\mu(s)$: we need, instead, to consider all the extended four-dimensional portions of spacetime over which the probability of finding the particle is nonvanishing. As a consequence, we have to specify a velocity distribution not along a unidimensional trajectory $\dot{x}^\mu = \dot{x}^\mu(s)$, but over the corresponding extended region, which in general spreads out to cover the whole spacetime: in this case a velocity field $\dot{x}^\mu = \dot{x}^\mu(\xi^\nu)$ defined over V_4 characterizes the motion of the particle.

The four equations $\dot{x}^\mu = \dot{x}^\mu(\xi^\nu)$, together with those $x^\mu = x^\mu(\xi^\nu)$ that relate the coordinates x^μ of M_8 to the coordinates ξ^μ chosen to parametrize V_4, constitute the set of eight parametric equations $x^A = x^A(\xi^\mu)$ needed to represent V_4 as a hypersurface in M_8; given any velocity field $\dot{x}^\mu(\xi^\nu)$ over V_4, corresponding to some particular dynamical situation, its embedding in phase space is thus determined.

Once the parametric equations $x^A(\xi)$ are specified, we have

$$dx^A = \frac{\partial x^A(\xi)}{\partial \xi^\nu}\, d\xi^\nu$$

and the spacetime metric $g_{\mu\nu}(\xi)$, locally induced on V_4 by the line element (1.1) of M_8, i.e., such that

$$d\tau^2 = g_{AB}\, dx^A\, dx^B = g_{\mu\nu}(\xi)\, d\xi^\mu\, d\xi^\nu \tag{2.3}$$

is then given by

$$g_{\mu\nu}(\xi) = g_{AB}\frac{\partial x^A}{\partial \xi^\mu}\frac{\partial x^B}{\partial \xi^\nu} = \eta_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\xi^\nu} + \frac{1}{m^2}\frac{\partial \dot{x}^\alpha}{\partial \xi^\mu}\frac{\partial \dot{x}^\beta}{\partial \xi^\nu}\right) \tag{2.4}$$

Even starting from a phase space M_8 with a flat metric, in the case of interacting particles, characterized by a velocity field $\dot{x}^\mu$ not trivially constant $(\partial \dot{x}^\mu/\partial \xi^\nu \neq 0)$, we have then an effective four-dimensional geometry which, in general, is curved [the curvature tensor corresponding to the metric (2.4) can be expressed, for instance, in terms of the well-known Gauss equation; see Eisenhart (1949)].

The effective spacetime curvature at the microscopic level and in the quantum regime is not absolute, therefore, but particle dependent: particles with different interactions are in general characterized by different velocity fields $\dot{x}^\mu(\xi^\nu)$, so that the intrinsic four-dimensional geometry they experience is described by different metric tensors.

In the classical limit $\hbar/mc \to 0$ the quantum contributions to the geometry vanish and the second term in the parentheses of equation (2.4) disappears; in this case the effective metric $g_{\mu\nu}$ may differ from the Minkowski one at most by a reparametrization, $x^\mu \to x^\mu(\xi^\nu)$, and the intrinsic flat nature of the macroscopic geometry thus remains unchanged.

3. A SIMPLE EXAMPLE: THE RINDLER VELOCITY FIELD

In the previous section we saw that, if the physical spacetime V_4 is regarded as a hypersurface in M_8 (i.e., in a manifold whose coordinates represent positions and velocities), the intrinsic metric on V_4 depends on the velocity field we define on it; that is, on the set of classical trajectories we use to cover the spacetime in order to describe motion at a microscopic level.

A constant velocity field $d\dot{x}^\mu = 0$ (corresponding to a free particle) obviously gives a trivial embedding which preserves spacetime flatness; in order to obtain quantum corrections to the microscopic curvature, we have to consider velocity fields with nonvanishing acceleration. The simplest example is the motion with constant proper acceleration: we consider then the portion of spacetime spanned by the world lines of uniformly accelerated observers

$$x = \frac{1}{\alpha} \cosh \alpha s, \qquad t = \frac{1}{\alpha} \sinh \alpha s \tag{3.1}$$

obtained by varying α and s according to the Rindler parametrization $\xi = 1/\alpha$, $\eta = \alpha s$ [in what follows we work, for simplicity, with a bidimensional spacetime, parametrized by (ξ, η), so that phase space is only four-dimensional]. The corresponding velocity field is

$$\dot{x} = \sinh \eta, \qquad \dot{t} = \cosh \eta \tag{3.2}$$

and the parametric equations for the embedding in M_8 are then

$$x^\mu(\xi, \eta) = (\xi \cosh \eta, \xi \sinh \eta), \qquad \dot{x}^\mu(\xi, \eta) = (\sinh \eta, \cosh \eta) \tag{3.3}$$

From equations (2.3) and (2.4) we obtain therefore that the Rindler line element $ds^2 = \xi^2\, d\eta^2 - d\xi^2$ is generalized as follows:

$$d\tau^2 = (\xi^2 - m^{-2})\, d\eta^2 - d\xi^2 \tag{3.4}$$

It should be mentioned that this quantum correction to the usual Rindler metric, though apparently very simple, leads to some interesting physical consequences. The first point to be stressed is that the horizon of this manifold is now given by $\xi = m^{-1}$, instead of $\xi = 0$; it is then represented in the (x, t) plane not by the null rays $x^2 = t^2$, but by the "maximal acceleration hyperbola" $x^2 - t^2 = m^{-2}$, corresponding to the world line of a uniformly accelerated particle with constant proper acceleration $\alpha = m$ [see equation (3.1)].

This is in agreement with the suggestion, previously discussed with different arguments (Caianiello, 1981, 1984; Caianiello *et al.*, 1982*a*), that in the context of quantum physics a natural limit should exist for the proper acceleration of a particle, fixed by its mass. Moreover, it is interesting to note that the replacement of the light cone by a hyperbola as the boundary of the Rindler spacetime provides automatically, as a consequence of the quantum corrections, the formal horizon regularization recently introduced *ad hoc* to quantize a string in an accelerated frame (De Vega and Sanchez, 1988).

Another important difference from the usual Rindler case is that the metric (3.4) describes a curved manifold. The nonvanishing scalar curvature is given by

$$R = -\frac{2}{m^2}(\xi^2 - m^{-2})^{-2} \tag{3.5}$$

and diverges for $\xi = m^{-1}$. Therefore in this case the horizon is a true physical curvature singularity and not just a removable coordinate singularity.

In order to investigate whether it is still possible to associate a temperature to an accelerated observer, one has to consider the vacuum state of a quantum field in the background metric (3.4). Even in first quantization, however, there are differences with respect to the Rindler manifold. Consider in fact the following coordinate transformation:

$$m\xi = \cosh\rho \quad (\rho \geq 0), \qquad m\xi = -\cosh\rho \quad (\rho \leq 0) \tag{3.6}$$

which changes the line element (3.4) in the conformally flat form

$$d\tau^2 = \frac{1}{m^2}\sinh^2\rho(d\eta^2 - d\rho^2) \tag{3.7}$$

The Klein-Gordon equation for a scalar particle of mass m, minimally coupled to gravity, becomes in this metric

$$\left(\frac{\partial^2}{\partial\eta^2} - \frac{\partial^2}{\partial\rho^2} + \sinh^2\rho\right)\phi(\eta, \rho) = 0 \tag{3.8}$$

Looking for solutions of the form $\phi = \psi(\rho)\exp(-i\omega\eta)$, we are led to a Schrödinger-like equation for $\psi(\rho)$, with an effective potential which tends to infinity in the limits $\rho = \pm\infty$: in this manifold we thus have a discrete energy spectrum, unlike the case of the Rindler spacetime.

To emphasize the differences from the continuum case, we can consider the high-acceleration regime $\alpha \to m$, which means $\rho \ll 1$ [see equation (3.6)]. By expanding the effective potential near the origin, one obtains that, to first order in ρ^2, equation (3.8) coincides formally with the energy eigenvalue equation of a harmonic oscillator. Therefore in this approximation the allowed value of ω are $\omega_n^2 = 2n+1$, with $n = 0, 1, 2, \ldots$, and the energy spectrum for a local observer with proper acceleration α, i.e., $E_n = \omega_n/\sqrt{g_{44}}$, is then given, in the limit $\alpha \to m$, by

$$E_n = \left[\frac{m^2\alpha}{m-\alpha}\left(n+\frac{1}{2}\right)\right]^{1/2} \tag{3.9}$$

[from equation (3.7) we have in fact, in the approximation considered, $g_{44} \simeq \rho^2 m^{-2}$, and, from (3.6), $\rho/2 \simeq m\alpha^{-1} - 1$].

4. CONCLUDING REMARKS

In this paper we started from the hypothesis that, at a microscopic level, the four-dimensional spacetime interval should be replaced by a generalized invariant interval defined in eight-dimensional phase space. The quantum corrections to the macroscopic geometry obtained in this context are weighted by the mass of the particle considered, m^{-2} [see equation (2.4)], and then should not be, in general, negligible like the usual general relativistic quantum effects, which, being proportional to the Newton constant, are weighted by the Planck mass, $M_p^{-2} \ll m^{-2}$.

Considering in particular quantum corrections to the Rindler metric, we have shown that uniformly accelerated particles should be characterized by a discrete energy spectrum in which, in the high-acceleration approximation ($\alpha \to m$), the levels corresponding to the square of the proper energy are equispaced, i.e., $E_n^2 \propto n$ (a similar behavior is also typical of the mass spectrum in dual resonance models and string theory (Rebbi, 1974), and was also obtained, though in a different context, in some of our previous work (Caianiello and Vilasi, 1981; Caianiello *et al.*, 1982*b*).

The main result of the geometric model we have considered is that the intrinsic spacetime geometry, in the quantum regime, is determined by the velocity field which describes its embedding in phase space. The microscopic spacetime intervals, therefore, are not absolute, but particle dependent: particles which are acted on by different interactions experience different acceleration fields, and thus are seen, by an external observer, as embedded in different four-dimensional geometries.

Among the possible consequences of this effect, we mention the possibility of providing a physical, and as formal, justification to geometric models for the confining aspects of strong interactions (Salam and Strathdee,

1978) and for the hadronization process (Gasperini, 1987*a*; Bediaga *et al.*, 1988, 1989), which are based on the representation of hadronic bags as "microuniverses." In fact, if the spacetime curvature, on a microscopic scale, is determined by acceleration, one can easily understand (Caianiello *et al.*, 1988) [without introducing new interactions with *ad hoc* coupling constants as in the work of Salam and Strathdee (1978)], why, within hadrons, a quark, being accelerated by color fields, is seen to interact with a curved geometry, while a lepton, which is not affected by strong interactions, lives in a flat spacetime and can escape freely. [The possibility of considering acceleration as the source of curvature for the hadronic bags was also considered in a previous work (Gasperini, 1988), but there interpreted as a thermal effect.]

Finally, we observe that, throughout this paper, we have discussed quantum corrections to the four-dimensional metric for particles accelerated by external fields in a flat background geometry. By using the same procedure we could as well consider, however, particles accelerated by gravity, i.e., by the geometry of a curved four-dimensional manifold, for example, according to the equations of geodesic deviation (Gasperini, 1987*b*). In this case the embedding of spacetime into phase space, determined by the velocity field corresponding to the gravity-induced trajectories, provides corrections to the given metric which can be interpreted as a "quantum back reaction," on the initial geometry, determined by the motion of particles in its background.

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